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Kernel of star-product for spin tomograms

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Abstract

The spin-state tomogram is presented as the modulus squared of the matrix element of the $SU(2)$-group irreducible representation. For the spin system, we established explicitly a realization of the set of operators defining for spin tomograms the star-product formalism in terms of irreducible tensors. On the set of spin tomograms, the delta-function and 'standard' and Moyal-like kernels of the tomogram star-product are calculated explicitly in terms of Clebsch–Gordan and Racah coefficients and matrix elements of the $SU(2)$ irreducible representation. In the limit of infinite spin, the spin tomogram is shown to become the tomogram of the harmonic-oscillator state.

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1. Introduction

Representations of Lie groups such as the $SU(2)$ group (see, e.g., [1, 2]) are used to describe the quantum states of particles with spin. The Lie group representations are important ingredients of gauge theories (see, e.g., [3–5]) and nonlinear equations related to gauge theory models. Here we discuss a new approach to consider the $SU(2)$-group irreducible representations within the framework of a tomographic star-product quantization procedure.

The tomographic map of spin states onto a probability distribution, which can be used to describe the states as an alternative to the density operator, was elaborated in [6, 7]. An analogous construction was suggested in [8–10]. There exist other maps of spin operators onto functions [11–13]. These maps can be considered within the framework of the star-product procedure [14]. The tomographic map of spin operators was shown in [15] to realize a new version of the star-product procedure. In [16] a general construction of the star-product procedure was formulated. The tomographic map related to the Heisenberg–Weyl group [17] was shown to be described by means of a specific star-product procedure and the kernel, which

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determines the star-product, was explicitly calculated [16]. Various aspects and properties of spin tomograms were studied in [18–20]. The relations of different tomographic approaches to spin states were considered in [21]. Application of spin tomography to the problem of a Bose–Einstein condensate was studied in [22]. The tomogram of the particle state described by the wavefunction dependent on the position x ($-\infty < x < \infty$) was expressed in terms of the wavefunction in [23]. Such states can be related to the Heisenberg–Weyl group representation. Though general formulae for the kernel that determines the star-product of spin tomograms were discussed (see, for example, [24]), the explicit final expression for the kernel was not found until now, as well as the expressions of spin tomograms in terms of spin wavefunctions.

The aim of this work is to obtain an explicit expression of the star-product kernel for spin tomograms in terms of Clebsch–Gordan and Racah coefficients. We also find explicit expressions for the delta-function for a set of spin tomograms. The tomograms of pure and mixed states are expressed in terms of the modulus squared of the matrix element of the $SU(2)$ irreducible representation. Since in the limit of infinite spin the $SU(2)$ group can be transformed to the Heisenberg–Weyl group (see, for example, [8, 21, 25, 26]), the tomograms of spin states must become the tomograms of the harmonic-oscillator states. In [27–30] the asymptotic expressions of matrix elements of the $SU(2)$ irreducible representation in the limit of large spin were found. Using these asymptotics we will show that spin tomograms become the harmonic-oscillator-state tomograms for coherent and Fock states. The results obtained are demonstrated by taking explicitly the examples of spin-1/2 and spin-1 cases.

The paper is organized as follows. In section 2, we review the standard properties of spin states and spin operators. In section 3, the reconstruction formula for a spin density operator is rederived. In section 4, the tomograms of pure and mixed spin states are obtained. In section 5, the star-product procedure for the spin tomogram is reviewed. In section 6, the delta-function for the spin tomogram is obtained. In section 7, the kernel of the spin-tomogram star-product is shown explicitly. In section 8, quantum evolution of the spin tomograms within the framework of a Moyal-like approach [11, 31] is studied. In section 9, we demonstrate the results for the cases $s = 1/2$ and $s = 1$. Asymptotics of spin tomograms and the relation of the asymptotics to the harmonic-oscillator tomograms are studied in section 10. Perspectives and conclusions are presented in section 11.

2. Review of the properties of spin states and spin-related operators

An approach to map spin operators onto functions on the Bloch sphere [15] exists. Following [6, 7] we treat the expression for an arbitrary observable acting on spin states in terms of measurable mean values of the observable in the state with given spin projection onto a given direction considered in a rotated reference frame.

Below we describe some standard operators used to discuss the properties of spin states.

For arbitrary values of spin, let the observable $\hat{A}^{(j)}$ be represented by the matrix in the standard basis of the angular momentum generators $\hat{J}_i$, $i = 1, 2, 3$, defined through

$$\hat{J}^2|jm\rangle = j(j+1)|jm\rangle \qquad \hat{J}_3|jm\rangle = m|jm\rangle \tag{1}$$

as

$$\hat{A}^{(j)} = \sum_{m=-j}^{j}\sum_{m'=-j}^{j} A^{(j)}_{mm'}\,|jm\rangle\langle jm'| \tag{2}$$

where

$$A^{(j)}_{mm'} = \langle jm|\,\hat{A}^{(j)}\,|jm'\rangle \qquad m = -j, -j+1, \ldots, j-1, j. \tag{3}$$

Let us consider the spin-j projector operator onto the m_1 component along the z-axis,

$$\hat{\Pi}_{m_1}^{(j)} = |jm_1\rangle\langle jm_1| \tag{4}$$

and the same projector in a rotated reference frame by an element g of $SU(2)$,

$$\hat{\Pi}_{m_1}^{(j)}(g) = R^\dagger(g)\hat{\Pi}_{m_1}^{(j)}R(g). \tag{5}$$

$R(g)$ is a rotation operator of the $SU(2)$ irreducible representation with spin j. Since the projectors play an important role in constructing the tomographic map, we present several different expressions for these operators. The projector can be given in an alternative form in terms of the Dirac (Kronecker) delta-function:

$$\hat{\Pi}_{m_1}^{(j)} = \delta(m_1 - \hat{J}_3). \tag{6}$$

The rotated projector can also be expressed in terms of the Dirac (Kronecker) delta-function as follows:

$$\hat{\Pi}_{m_1}^{(j)}(g) = \delta(m_1 - R^\dagger(g)\hat{J}_3R(g)) \tag{7}$$

or in integral form

$$\hat{\Pi}_{m_1}^{(j)}(g) = \frac{1}{2\pi}\int_0^{2\pi} \exp[\mathrm{i}(m_1 - R^\dagger(g)\hat{J}_3R(g))\varphi]\,\mathrm{d}\varphi. \tag{8}$$

Another form of the rotated projector is given by the expression

$$\hat{\Pi}_{m_1}^{(j)}(g) = \sum_{m_1'm_2'} D_{m_1m_2'}^{(j)*}(\alpha,\beta,\gamma)D_{m_1m_1'}^{(j)}(\alpha,\beta,\gamma)|jm_2'\rangle\langle jm_1'|. \tag{9}$$

The matrix elements $D_{m_1m_1'}^{(j)}(\alpha,\beta,\gamma)$ (Wigner D-functions) are the matrix elements of the operator

$$R(g) = \mathrm{e}^{-\mathrm{i}\alpha\hat{J}_3}\,\mathrm{e}^{-\mathrm{i}\beta\hat{J}_2}\mathrm{e}^{-\mathrm{i}\gamma\hat{J}_3} \tag{10}$$

of the $SU(2)$ group representation (g is an element of the $SU(2)$ group parametrized by the Euler angles). The matrix elements have the known explicit form

$$D_{m'm}^{(j)}(\alpha,\beta,\gamma) = \mathrm{e}^{-\mathrm{i}m'\alpha}d_{m'm}^{(j)}(\beta)\,\mathrm{e}^{-\mathrm{i}m\gamma} \tag{11}$$

where

$$d_{m'm}^{(j)}(\beta) = \sum_s \frac{(-1)^s\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{s!(j-m'-s)!(j+m-s)!(m'-m+s)!} \times \left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2s}\left(-\sin\frac{\beta}{2}\right)^{m'-m+2s}. \tag{12}$$

It is convenient to introduce the irreducible tensor operator for the $SU(2)$ group:

$$\hat{T}_{LM}^{(j)} = \sum_{m_1,m_2=-j}^{j} (-1)^{j-m_1}\langle jm_2;\, j-m_1|LM\rangle|jm_2\rangle\langle jm_1|. \tag{13}$$

Some examples of these tensors for $j = 1/2$ and $j = 1$ are given in section 9.

The irreducible tensors have the known properties (see [13, 26, 32])

$$\mathrm{Tr}\big(\hat{T}_{L_1M_1}^{(j)\dagger}\hat{T}_{L_2M_2}^{(j)}\big) = \delta_{L_1L_2}\delta_{M_1M_2} \tag{14}$$

$$\mathrm{Tr}\big(\hat{T}_{L_1M_1}^{(j)}\hat{T}_{L_2M_2}^{(j)}\hat{T}_{LM}^{(j)}\big) = (-1)^{L_1+L_2+L-2j}\begin{pmatrix} L_1 & L_2 & L\\ M_1 & M_2 & M\end{pmatrix}\begin{Bmatrix} L_1 & L_2 & L\\ j & j & j\end{Bmatrix} \times\sqrt{(2L_1+1)(2L_2+1)(2L+1)}. \tag{15}$$

The operator $|jm\rangle\langle jm'|$ can be expressed in terms of the irreducible tensors as follows:

$$|jm\rangle\langle jm'| = \sum_{L=0}^{2j}\sum_{M=-L}^{L}(-1)^{j-m'}\langle jm;\, j-m'|LM\rangle\hat{T}_{LM}^{(j)}. \qquad (16)$$

This means that the irreducible tensors form a basis in the linear space of operators acting on the unitary space of the $SU(2)$ irreducible representation.

3. Tomogram and reconstruction formula

Following [6, 7, 15, 16, 20, 21] we introduce the tomogram of the observable $\hat{A}^{(j)}$:

$$\begin{aligned} w(m_1,\beta,\gamma) &= \mathrm{Tr}[\hat{A}^{(j)}R^{\dagger}(g)|jm_1\rangle\langle jm_1|R(g)] \\ &= \sum_{m_1'=-j}^{j}\sum_{m_2'=-j}^{j} D_{m_1m_1'}^{(j)}(\alpha,\beta,\gamma)A_{m_1'm_2'}^{(j)}D_{m_1m_2'}^{(j)*}(\alpha,\beta,\gamma). \end{aligned} \qquad (17)$$

In view of the structure of formula (17), the tomogram depends only on two Euler angles, i.e. the tomogram depends on the spin projection and a point on the Bloch sphere.

The tomogram can be presented in another form using the Kronecker delta-function, which is the general form for the tomograms of arbitrary observables suggested in [33]:

$$w(m_1,\beta,\gamma) = \mathrm{Tr}\hat{A}^{(j)}\delta(m_1 - R^{\dagger}(g)\hat{J}_3R(g)). \qquad (18)$$

It is obvious that the tomogram of the identity operator equals unity.

To derive the inverse of (17), we multiply it by the Wigner D-function $D_{\mu'm'}^{j'}(\alpha,\beta,\gamma)$, and integrate over the volume element of the $SU(2)$ group, i.e.,

$$\int \mathrm{d}\Omega w(m_1,\beta,\gamma)D_{\mu'm'}^{j'}(\alpha,\beta,\gamma) = \sum_{m_1'm_2'}\langle j'm';\, jm_1'|jm_2'\rangle\langle j'\mu';\, jm_1|jm_1\rangle\frac{8\pi^2}{2j+1}A_{m_1'm_2'}^{j} \qquad (19)$$

where the known property of the Wigner D-functions ($D(\alpha,\beta,\gamma)\equiv D(\Omega)$)

$$\int \mathrm{d}\Omega D_{m_3'm_3}^{j_3*}(\Omega)D_{m_2'm_2}^{j_2}(\Omega)D_{m_1'm_1}^{j_1}(\Omega) = \frac{8\pi^2}{2j_3+1}\langle j_1m_1;\, j_2m_2|j_3m_3\rangle\langle j_1m_1';\, j_2m_2'|j_3m_3'\rangle \qquad (20)$$

was used.

In view of the symmetry relations and properties of the Clebsch–Gordan coefficients, we have that

$$\begin{aligned} &\langle j'm';\, jm_1'|jm_2'\rangle\langle j'\mu';\, jm_1|jm_2\rangle \\ &\quad = (-1)^{j+m_1+j+m_1'}\frac{2j+1}{2j'+1}\delta_{\mu'0}\langle jm_1;\, j-m_1|j'0\rangle\langle jm_2';\, j-m_1'|j'm'\rangle. \end{aligned} \qquad (21)$$

Using the orthonormality property of the Clebsch–Gordan coefficients

$$\sum_{m_1=-j}^{j}\langle jm_1;\, j-m_1|j'0\rangle\langle jm_1;\, j-m_1|j'0\rangle = 1$$

we have that

$$\begin{aligned} &\sum_{m_1}\frac{2j'+1}{8\pi^2}\langle jm_1;\, j-m_1|j'0\rangle\int \mathrm{d}\Omega(-1)^{j+m_1}w(m_1,\beta,\gamma)D_{0m'}^{j'}(\Omega) \\ &\quad = \sum_{m_1',m_2'=-j}^{j}(-1)^{j+m_1'}\langle jm_2';\, j-m_1'|j'm'\rangle A_{m_1'm_2'}^{(j)}. \end{aligned} \qquad (22)$$

Multiplying this equation by $\langle j\mu_2;\ j\mu_1|j'm'\rangle$ and summing over the indices j' and m', we arrive at the result

$$A^{(j)}_{\mu_1\mu_2} = \sum_{j'=0}^{2j}\sum_{m'=-j'}^{j'}\sum_{m_1=-j}^{j}(-1)^{m_1-\mu_1}\langle jm_1;\ j-m_1|j'0\rangle\langle j\mu_1;\ j-\mu_2|j'm'\rangle \times \frac{(2j'+1)}{8\pi^2}\int \mathrm{d}\Omega\, w(m_1,\beta,\gamma)D^{j'}_{0-m'}(\alpha,\beta,\gamma). \tag{23}$$

Using equations (2) and (16) we can write the observable operator $\hat{A}^{(j)}$ in terms of unitary irreducible tensors as follows:

$$\hat{A}^{(j)} = \sum_{\mu_1,\mu_2=-j}^{j}\sum_{L=0}^{2j}\sum_{M=-L}^{L}(-1)^{j-\mu_2}\langle j\mu_1;\ j-\mu_2|LM\rangle\hat{T}^{(j)}_{LM}A^{(j)}_{\mu_1\mu_2}. \tag{24}$$

Substituting $A^{(j)}_{\mu_1\mu_2}$ into (24), in view of the orthonormality of the Clebsch–Gordan coefficients, we get the observable in terms of its corresponding tomogram:

$$\hat{A}^{(j)} = \sum_{L=0}^{2j}\sum_{M=-L}^{L}\sum_{m=-j}^{j}(-1)^{j-m+M}\frac{2L+1}{8\pi^2}\langle jm;\ j-m|L0\rangle \times \left(\int \mathrm{d}\Omega\, w(m,\beta,\gamma)D^{L}_{0-M}(\alpha,\beta,\gamma)\right)\hat{T}^{(j)}_{LM}. \tag{25}$$

The density operator $\hat{\rho}$ can be expanded in terms of irreducible tensors (13) as follows:

$$\hat{\rho} = \sum_{L=0}^{2j}\sum_{M=-L}^{L}(-1)^{M}\frac{2L+1}{8\pi^2}\int \mathrm{d}\Omega D^{(L)}_{0-M}(\alpha,\beta,\gamma) \times \sum_{m=-j}^{j}(-1)^{j-m}w(m,\beta,\gamma)\langle jm;\ j-m|L0\rangle\hat{T}^{(j)}_{LM}. \tag{26}$$

4. Tomogram of pure and mixed states

If one considers the tomogram of the density operator

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi|$$

of the pure state $|\psi\rangle$, the tomogram reads

$$w_\psi(m_1,\beta,\gamma) = |\langle jm_1|R(g)|\psi\rangle|^2. \tag{27}$$

This means that the tomogram is the modulus squared of the matrix element of matrix $R(g)$ of the $SU(2)$ irreducible representation between the states $|\psi\rangle$ and $|jm_1\rangle$. If one has the density operator of mixed state written in terms of its eigenvalues $p_i \geqslant 0$, $\sum_i p_i = 1$ and eigenvectors $|\psi_i\rangle$ in the form

$$\hat{\rho} = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{28}$$

the state tomogram can also be expressed in terms of the modulus squared of the matrix elements of the rotation group

$$w(m_1,\beta,\gamma) = \sum_i p_i|\langle jm_1|R(g)|\psi_i\rangle|^2. \tag{29}$$

The above two formulae are an extension of the expression of the state tomogram found in [23] for the position observable in terms of the wavefunction to the case of spin-state tomograms.

For example, if the state $|\psi\rangle$ is the pure state $|j\mu\rangle$, its tomogram reads

$$w_{j\mu}(m_1, \beta, \gamma) = \left|d^{(j)}_{m_1\mu}(\beta)\right|^2 \tag{30}$$

where $d^{(j)}_{m_1M}(\beta)$ is given by (12). The tomogram depends only on the Euler angle β.

The existence of inverse relation (26) means that the modulus of the matrix element of the $SU(2)$ irreducible representation determines the phase of the matrix element.

5. Star-product for spin tomograms

In this section, we describe the relation of the above construction to the star-product procedure in the form of [15, 16].

In quantum mechanics, observables are described by operators acting on a Hilbert space of states. Given a Hilbert space $\mathcal{H}$ of spin states and an operator $\hat{A}$ acting on this space, let us suppose that we have a set of operators $\hat{U}(\mathbf{x})$ acting on $\mathcal{H}$ in terms of a four-dimensional vector $\mathbf{x} = (m, \alpha, \beta, \gamma)$, where the first component is a spin projection and the other three Euler angles parametrize an element of the $SU(2)$ group. We construct the c-number function $f_{\hat{A}}(\mathbf{x})$ (called the symbol of operator $\hat{A}$) using the definition

$$f_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}[\hat{A}\hat{U}(\mathbf{x})]. \tag{31}$$

Let us suppose that relation (31) has an inverse, i.e., there exists a set of operators $\hat{D}(\mathbf{x})$ acting on Hilbert space such that

$$\hat{A} = \int f_{\hat{A}}(\mathbf{x})\hat{D}(\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \int \mathrm{d}\mathbf{x} = \sum_{m_1=-j}^{j} \int \mathrm{d}\Omega. \tag{32}$$

That is, in this formula, $\int \mathrm{d}\mathbf{x}$ means integration over continuous variables and summation over the discrete component. Then we consider relations (31) and (32) as relations determining the invertible map of the operator $\hat{A}$ onto function $f_{\hat{A}}(\mathbf{x})$.

The most important property of the map is the existence of the associative product (star-product) of symbols.

We introduce the product (star-product) of two symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$ corresponding to two operators $\hat{A}$ and $\hat{B}$ by the relations

$$f_{\hat{A}\hat{B}}(\mathbf{x}) = f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) := \mathrm{Tr}[\hat{A}\hat{B}\hat{U}(\mathbf{x})]. \tag{33}$$

Since the standard product of operators on a Hilbert space is an associative product, i.e.

$$\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$$

it is obvious that formula (33) defines an associative product for the functions, i.e.

$$f_{\hat{A}}(\mathbf{x}) * (f_{\hat{B}}(\mathbf{x}) * f_{\hat{C}}(\mathbf{x})) = (f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x})) * f_{\hat{C}}(\mathbf{x}). \tag{34}$$

In view of (31), the commutation relation of two operators

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{35}$$

is mapped onto the Poisson bracket $f_{\hat{C}}(\mathbf{x})$ of two symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$ by means of the formula

$$f_{\hat{C}}(\mathbf{x}) = \{f_{\hat{A}}(\mathbf{x}), f_{\hat{B}}(\mathbf{x})\}_* = \mathrm{Tr}[[\hat{A}, \hat{B}]\hat{U}(\mathbf{x})]. \tag{36}$$

One can express the operators determining the star-product of tomographic symbols in terms of irreducible tensors. By comparing the formulae defining the generic symbol of operators (31) and its inverse (32) with the formulae defining the observable tomogram (18) and its inverse (25), one can find the operators $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$ explicitly. It is obvious that the operators $\hat{U}(\mathbf{x}) \equiv \hat{U}(m, \Omega)$ and $\hat{D}(\mathbf{x}) \equiv \hat{D}(m, \Omega)$ can be expressed as follows:

$$\hat{U}(m, \Omega) = \sum_{L=0}^{2j} \sum_{M=-L}^{L} (-1)^{j-m+M} \langle jm; j-m|L0\rangle D_{0-M}^{L}(\alpha, \beta, \gamma) \hat{T}_{LM}^{(j)} \tag{37}$$

$$\hat{D}(m, \Omega) = \sum_{L=0}^{2j} \sum_{M=-L}^{L} (-1)^{j-m+M} \frac{2L+1}{8\pi^2} \langle jm; j-m|L0\rangle D_{0-M}^{L}(\alpha, \beta, \gamma) \hat{T}_{LM}^{(j)}. \tag{38}$$

These formulae are convenient to calculate explicitly the kernel of the star-product of tomographic symbols.

6. Delta-function on the tomogram set

In view of the properties of operators $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$, the following equality for the spin tomogram must exist:

$$|\langle jm|R(g)|\psi\rangle|^2 = \sum_{m'=-j}^{j} \int \mathrm{d}\Omega' \, \mathrm{Tr}[\hat{D}(m', \Omega')\hat{U}(m, \Omega)] |\langle jm'|R(g')|\psi\rangle|^2 \tag{39}$$

which is valid for an arbitrary spin state $|\psi\rangle$. This equality means that on the set of spin tomograms $w^{(j)}(m, \Omega)$ the trace of the product of the operators $\hat{D}(\mathbf{x})$ and $\hat{U}(\mathbf{x})$ plays the role of an analogue of the Dirac delta-function and we denote it as

$$\delta(m, m', \Omega, \Omega') = \mathrm{Tr}[\hat{D}(m', \Omega')\hat{U}(m, \Omega)]. \tag{40}$$

In fact, it is the kernel of the unity operator on the set of spin tomograms. For spin j, one can write explicitly this kernel or delta-function:

$$\mathrm{Tr}[\hat{D}(m', \Omega')\hat{U}(m, \Omega)] = \sum_{L=0}^{2j} (-1)^{m-m'} \frac{2L+1}{8\pi^2} \langle jm; j-m|L0\rangle \langle jm'; j-m'|L0\rangle P_L(\cos\theta)$$

where $P_L(x)$ is the Legendre polynomial of order L and θ is the angle between directions (β, γ) and (β', γ') given by

$$\cos\theta = \cos\beta \cos\beta' + \sin\beta \sin\beta' \cos(\gamma - \gamma').$$

7. Kernel of standard star-product

Using formulae (31) and (32), one can write down a composition rule for two symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$, which determines the star-product of these symbols. The composition rule is described by the formula

$$f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) = \int f_{\hat{A}}(\mathbf{x}'') f_{\hat{B}}(\mathbf{x}') K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}''. \tag{41}$$

The kernel in the integral of (41) is determined by the trace of the product of the basic operators, which we use to construct the map [16]:

$$K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \mathrm{Tr}[\hat{D}(\mathbf{x}'')\hat{D}(\mathbf{x}')\hat{U}(\mathbf{x})]. \tag{42}$$

Within this framework, according to (4), (5) and (7), we have two equivalent expressions for the operator $\hat{U}(\mathbf{x})$:

$$\hat{U}(\mathbf{x}) = \delta(m_1 - R^\dagger(g)\hat{J}_3 R(g)) = R(g)^\dagger|jm_1\rangle\langle jm_1|R(g) \tag{43}$$

or due to the structure of this equation

$$\hat{U}(\mathbf{x}) = \delta(m_1 - \mathbf{n}\cdot\hat{\mathbf{J}}) \qquad \mathbf{n} = (\sin\beta\cos\gamma, \sin\beta\sin\gamma, \cos\beta)\,. \tag{44}$$

The dual operator reads

$$\hat{D}(\mathbf{x}) = \sum_{L=0}^{2j}\sum_{M=-L}^{L}(-1)^{j-m+M}\frac{2L+1}{8\pi^2}D^{(L)}_{0-M}(\alpha,\beta,\gamma)\langle jm;\, j-m|L0\rangle\hat{T}^{(j)}_{LM} \tag{45}$$

where $\hat{T}^{(j)}_{LM}$ is given in equation (13).

Inserting the expressions for the operators $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$ in (42) and using the properties of irreducible tensors (14) and (15), one can calculate explicitly the kernel of the spin star-product, which has the form

$$\begin{aligned}K(\mathbf{x}_2,\mathbf{x}_1,\mathbf{x}) &\equiv K(m_2,\Omega_2,m_1,\Omega_1,m,\Omega)\\ &= (-1)^{j-m-m_1-m_2}\sum_{L=0}^{2j}\sum_{L_1=0}^{2j}\sum_{L_2=0}^{2j}\frac{(2L_1+1)(2L_2+1)}{64\pi^4}\\ &\quad\times\langle jm;\, j-m|L0\rangle\langle jm_1;\, j-m_1|L_10\rangle\langle jm_2;\, j-m_2|L_20\rangle\\ &\quad\times\sum_{M=-L}^{L}\sum_{M_1=-L_1}^{L_1}\sum_{M_2=-L_2}^{L_2}(-1)^{L+L_1+L_2}\sqrt{(2L+1)(2L_1+1)(2L_2+1)}\\ &\quad\times\begin{Bmatrix}L_1 & L_2 & L\\ j & j & j\end{Bmatrix}\begin{pmatrix}L_1 & L_2 & L\\ M_1 & M_2 & M\end{pmatrix}D^{(L)}_{0-M}(\Omega)D^{(L_1)}_{0-M_1}(\Omega_1)D^{(L_2)}_{0-M_2}(\Omega_2)\,.\end{aligned} \tag{46}$$

Thus we obtained the explicit expression for the star-product kernel of spin tomograms in terms of Clebsch–Gordan coefficients, $6j$-symbols (Racah coefficients) and Wigner D-functions.

8. Evolution of tomograms and Moyal-like kernel

In quantum mechanics, the evolution of nonexplicitly time-dependent observables $\hat{A}$ can be described by the Heisenberg equation of motion

$$\dot{\hat{A}} = \mathrm{i}[\hat{H},\hat{A}] \qquad (\hbar = 1) \tag{47}$$

where $\hat{H}$ is the system Hamiltonian. This equation can be rewritten in terms of symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{H}}(\mathbf{x})$ in the form

$$\dot{f}_{\hat{A}}(\mathbf{x},t) = \mathrm{i}\{f_{\hat{H}}(\mathbf{x},t), f_{\hat{A}}(\mathbf{x},t)\}_* \tag{48}$$

where

$$f_{\hat{H}}(\mathbf{x}) = \mathrm{Tr}[\hat{H}\hat{U}(\mathbf{x})] \tag{49}$$

corresponds to the Hamiltonian, and with the Poisson bracket defined by equation (36) using the star-product given by equation (33).

The quantum evolution equation for the symbol of the observable can be presented in the form of an integral equation.

The kernel of the star-product obtained in the previous section determines the kernel, which provides the integral form of the quantum evolution equation (47). The kernel corresponding to the Poisson bracket term reads

$$
\begin{aligned}
K_m(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}) &\equiv K(m_2, \Omega_2, m_1, \Omega_1, m, \Omega) - K(m_1, \Omega_1, m_2, \Omega_2, m, \Omega) \\
&= (-1)^{j-m-m_1-m_2} \sum_{L=0}^{2j} \sum_{L_1=0}^{2j} \sum_{L_2=0}^{2j} [1 - (-1)^{L+L_1+L_2}] \frac{(2L_1+1)(2L_2+1)}{64\pi^4} \\
&\quad \times \langle jm;\, j-m|L0\rangle \langle jm_1;\, j-m_1|L_10\rangle \langle jm_2;\, j-m_2|L_20\rangle \\
&\quad \times \sum_{M=-L}^{L} \sum_{M_1=-L_1}^{L_1} \sum_{M_2=-L_2}^{L_2} (-1)^{L+L_1+L_2} \sqrt{(2L+1)(2L_1+1)(2L_2+1)} \\
&\quad \times \begin{Bmatrix} L_1 & L_2 & L \\ j & j & j \end{Bmatrix} \begin{pmatrix} L_1 & L_2 & L \\ M_1 & M_2 & M \end{pmatrix} D^{(L)}_{0-M}(\Omega) D^{(L_1)}_{0-M_1}(\Omega_1) D^{(L_2)}_{0-M_2}(\Omega_2)\,.
\end{aligned} \tag{50}
$$

For the evolution of the magnetic moment moving in the magnetic field, the star-product formalism was used in [15]. The kernel (50) gives the possibility of studying the quantum evolution of spin states for large values of spin j.

9. Examples of $j = 1/2$ and $j = 1$

Next we give for the $SU(2)$-group irreducible representations the explicit form of the operators $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$, which define the star-product of spin tomograms for $j = 1/2$ and $j = 1$ cases. First we give the irreducible tensors for $j = 1/2$. These can be written in terms of the Pauli matrices $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$ and the identity matrix $\boldsymbol{I}_{2\times 2}$ (we omit hats)

$$
\begin{aligned}
&T^{(1/2)}_{00} = \tfrac{1}{\sqrt{2}} \boldsymbol{I}_{2\times 2} \\
&T^{(1/2)}_{11} = -\tfrac{1}{\sqrt{2}} \boldsymbol{\sigma}_+ \quad T^{(1/2)}_{10} = \tfrac{1}{\sqrt{2}} \boldsymbol{\sigma}_z \qquad T^{(1/2)}_{1-1} = \tfrac{1}{\sqrt{2}} \boldsymbol{\sigma}_-
\end{aligned}
$$

where $\boldsymbol{\sigma}_\pm = \boldsymbol{\sigma}_x \pm \mathrm{i}\boldsymbol{\sigma}_y$.

For the case $j = 1$, they can be written in terms of the matrix representations of the angular momentum operators $\boldsymbol{J}_\pm$ and $\boldsymbol{J}_0$ and their products:

$$
\begin{aligned}
&T^{(1)}_{00} = \tfrac{1}{\sqrt{3}} \boldsymbol{I}_{3\times 3} \\
&T^{(1)}_{11} = -\tfrac{1}{2} \boldsymbol{J}_+ && T^{(1)}_{10} = \tfrac{1}{\sqrt{2}} \boldsymbol{J}_0 && T^{(1)}_{1-1} = \tfrac{1}{2} \boldsymbol{J}_- \\
&T^{(1)}_{22} = \tfrac{1}{2} \boldsymbol{J}_+^2 && T^{(1)}_{21} = -\tfrac{1}{2} (\boldsymbol{J}_+ \boldsymbol{J}_0 + \boldsymbol{J}_0 \boldsymbol{J}_+) && T^{(1)}_{20} = \sqrt{6}\big(\tfrac{1}{2} \boldsymbol{J}_0^2 - \tfrac{1}{3} \boldsymbol{I}_{3\times 3}\big) \\
&T^{(1)}_{2-1} = \tfrac{1}{2} (\boldsymbol{J}_- \boldsymbol{J}_0 + \boldsymbol{J}_0 \boldsymbol{J}_-) && T^{(1)}_{2-2} = \tfrac{1}{2} \boldsymbol{J}_-^2 .
\end{aligned}
$$

By means of expressions (37) and (38), we get for $j = 1/2$ that

$$
\hat{U}(\mathbf{x}) = \tfrac{1}{2} \boldsymbol{I}_{2\times 2} + m \boldsymbol{F}(\beta, \gamma) \tag{51}
$$

$$
\hat{D}(\mathbf{x}) = \frac{1}{8\pi^2} \left(\frac{1}{2} \boldsymbol{I}_{2\times 2} + 3m \boldsymbol{F}(\beta, \gamma) \right) \tag{52}
$$

where we defined the matrix

$$
\boldsymbol{F}(\beta, \gamma) = \begin{pmatrix} \cos\beta & -\mathrm{e}^{\mathrm{i}\gamma} \sin\beta \\ -\mathrm{e}^{-\mathrm{i}\gamma} \sin\beta & -\cos\beta \end{pmatrix}. \tag{53}
$$

For $j = 1$ we write the corresponding results in terms of the Wigner D-functions and the irreducible tensors, i.e.

$$\hat{U}(\mathbf{x}) = \frac{1}{\sqrt{3}}T_{00}^{(1)} + \sum_M (-1)^M \frac{m}{\sqrt{2}} D_{0-M}^{(1)}(\Omega) T_{1M}^{(1)} + \sum_M (-1)^M \frac{3m^2-2}{\sqrt{6}} D_{0-M}^{(2)}(\Omega) T_{2M}^{(1)} \tag{54}$$

$$\hat{D}(\mathbf{x}) = \frac{1}{8\pi^2}\left(\frac{1}{\sqrt{3}}T_{00}^{(1)} + \sum_M (-1)^M \frac{3m}{\sqrt{2}} D_{0-M}^{(1)}(\Omega) T_{1M}^{(1)} + \sum_M (-1)^M \frac{15m^2-10}{\sqrt{6}} D_{0-M}^{(2)}(\Omega) T_{2M}^{(1)}\right). \tag{55}$$

With these explicit expressions for $\hat{U}(\mathbf{x})$ and $\hat{D}(\mathbf{x})$ operators, it is straightforward to construct the tomogram of any physical observable in the Hilbert space for $j = 1/2$ and $j = 1$.

10. Asymptotic spin tomograms

In this section, we consider the behaviour of spin tomograms for pure states in the limit of high values of spin j.

Since the tomogram of a pure spin state $|jm_0\rangle$ is expressed in terms of the matrix element of the $SU(2)$-group irreducible representation, we use the known asymptotics of these elements for $j \to \infty$ studied in [27–30]. For example, there exists the asymptotics of the matrix elements of the $SU(2)$-group irreducible representations expressed in terms of the Hermite polynomials and one has the explicit relation of the form [30]

$$d_{mm'}^{(j)}(\beta) = (-1)^{j-m'}(\pi j \sin^2\beta)^{-1/4}[2^{j-m'}(j-m')!]^{-1/2} \times \exp[-2j\sin^2\beta] H_{j-m'}\left(\frac{m - j\cos\beta}{\sqrt{j}\sin\beta}\right). \tag{56}$$

The formula is similar to the expression for the wavefunction of the $(j - m')$th level of the harmonic oscillator. For $j = m'$, one has

$$d_{mj}^{(j)}(\beta) = (\pi j \sin^2\beta)^{-1/4}\exp[-2j\sin^2\beta] \tag{57}$$

a Gaussian which is identical to the wavefunction of the ground state of the harmonic oscillator. Correspondingly, the spin tomograms are given by the relations

$$w^{(j)}(m,\beta) = (\pi j \sin^2\beta)^{-1/2}[2^{j-m'}(j-m')!]^{-1} \times \exp[-2j\sin^2\beta] H_{j-m'}^2\left(\frac{m - j\cos\beta}{\sqrt{j}\sin\beta}\right) \tag{58}$$

and for $m' = j$ one has the tomogram asymptotics

$$w^{(j)}(m,\beta) = (\pi j \sin^2\beta)^{-1/2}\exp[-2j\sin^2\beta]. \tag{59}$$

One can see that these spin tomograms are identical to the tomograms of the energy level states of the harmonic oscillator $w(X,\mu,\nu)$ found in [34]. Thus, for the coherent state $|\alpha\rangle$ of the harmonic oscillator, the tomogram $w_\alpha(X,\mu,\nu)$, where μ, ν are real parameters and X is the position measured in a reference frame in the phase space of the oscillator labelled by these two parameters, reads

$$w_\alpha(X,\mu,\nu) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\exp\left[-\frac{\left(X - \sqrt{2}\,\mathrm{Re}\,\alpha\mu - \sqrt{2}\,\mathrm{Im}\,\alpha\nu\right)^2}{\mu^2+\nu^2}\right]. \tag{60}$$

The tomogram of the oscillator's ground state corresponds to $\alpha = 0$ and it is

$$w_0(X,\mu,\nu) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)}}\exp\left[-\frac{X^2}{\mu^2+\nu^2}\right]. \tag{61}$$

The tomogram of the nth energy level of the harmonic oscillator reads

$$w_n(X, \mu, \nu) = w_0(X, \mu, \nu)\frac{1}{2^n n!}H_n^2\left(\frac{X^2}{\mu^2 + \nu^2}\right). \quad (62)$$

One can see that the asymptotics of the $\left|d^{(j)}_{mm'}(\beta)\right|^2$ in (6) coincides with the tomogram (62) provided the following replacements are done:

$$m - j\cos\beta \to X \qquad j\sin^2\beta \to \mu^2 + \nu^2.$$

The tomogram asymptotics (59) coincides with the tomogram of the oscillator's coherent state provided the term $j\cos\beta$ is interpreted as the quadrature mean $\langle x\rangle$. We have checked by computer the validity of approximation (58) for the $d^{(j)}_{mm'}$ function. For $j = 5 \times 10^5$ and $\beta = \pi/2$, the maximum difference is $\Delta \sim 10^{-2}$ where

$$\Delta = \frac{\left|\left(d^{(j)}_{mm'}(\beta)\right)^2 - w^{(j)}(m, \beta)\right|}{|w^{(j)}(m, \beta)|}.$$

It is very small for all values of m and of the given order only for $m \approx -j$.

11. Conclusions

To conclude, we summarize the results of our study.

We have shown that the spin tomogram of the pure spin state with fixed spin projection is determined by the modulus squared of the matrix element of the $SU(2)$-group irreducible representation. This means that the modulus of the matrix element of the irreducible representation determines the phase of the matrix element (up to a constant factor). The new result of this work is the explicit expressions for the kernels which determine the delta-function for the spin-tomographic symbols and the star-product of the symbols, respectively.

Using the relation of Heisenberg–Weyl and $SU(2)$ groups in the limit of high spins, we calculate spin tomograms in this limit. We have shown that, for large spin values, the spin tomograms become the tomograms of the harmonic-oscillator states.

The approach developed demonstrates that one can use the standard probability distribution to describe the quantum spin state instead of spinors or density matrices. The tomogram construction resembles a finite-dimensional C^* operator algebra representation but the correspondence needs better clarification. The product of tomographic symbols of the spin operators is determined by the kernel expressed in terms of the standard ingredients of $SU(2)$-representation theory which are Wigner D-functions, Clebsch–Gordan coefficients and $6j$-symbols (Racah coefficients).

The new approach to the description of spinors by means of probability distributions can be applied to gauge theory models and related nonlinear equations based on the $SU(2)$ symmetry. Also the approach can be extended to other Lie group representations.

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